Math 565: Functional Analysis
Lecture 19

Obs. When $Y := \mathbb{C}$, then $B(X, \mathbb{C}) = X^*$ and the strong top is the weak* on X^*. Also, the weak top on $B(X, \mathbb{C})$ is still the weak* top on X^* just becaue the norm and weak coincide on $\mathbb{C}$.

Caution. Weak top on X^* (i.e. coming from evaluating against functionals in X^{**}) is stronger than the weak top on $B(X, \mathbb{C}) = X^*$ because the latter is the weak* top on X^*, which is strictly weaker than weak top on X^* when X is Banach and not reflexive.

Prop. Let X, Y be normed v.s. and $D \subseteq X$ be dense. Let $\mathcal{B} \subseteq B(X, Y)$ be a norm-bdd set, so $\sup_{T \in \mathcal{B}} \|T\| \leq r < \infty$ for some $r > 0$. Then the subspace top on $\mathcal{B}$ of the strong top $B(X, Y)$ is generated by the functions $T \mapsto Td$, where $d \in D$.
In other words, if $(T_i)_{i \in I} \subseteq \mathcal{B}$ is a net and $T \in \mathcal{B}$ then $T_i \to_s T \iff T_i d \to Td$ for all $d \in D$.

Remark. The boundedness of $\mathcal{B}$ is required to ensure the convergent net is bdd because unlike sequences, convergent nets need not be bounded.

Proof via open sets. Since the strong top on $\mathcal{B}$ is generated by the functions $T \mapsto Tx$, hence by sets $[x \mapsto U] := \{T \in \mathcal{B} : Tx \in U\}$ where $x \in X$ and $U \subseteq Y$ norm-open, it is enough to fix $T_0 \in [x \mapsto U]$ and show that $\exists V \subseteq Y$ norm-open and $d \in D$ such that $T_0 \in [d \mapsto V] \subseteq [x \mapsto U]$. Since $T_0 \in [x \mapsto U]$, we have $T_0 x \in U$, so $B_\varepsilon(T_0 x) \subseteq U$ for some $\varepsilon > 0$. Set $V := B_{\varepsilon/2}(T_0 x)$ and take $d \in D$ s.t. $\|x - d\| < \frac{\varepsilon}{2r}$.
Note that

$$T \in [d \mapsto B_{\varepsilon/2}(T_0 x)] \iff \|Td - T_0 x\| < \varepsilon/2 \qquad (1)$$

$$T \in [x \mapsto B_\varepsilon(T_0 x)] \iff \|Tx - T_0 x\| < \varepsilon. \qquad (2)$$

We show that $T_0 \in [d \mapsto B_{\varepsilon/2}(T_0 x)] \subseteq [x \mapsto B_\varepsilon(T_0 x)]$.

Note that $\|T_0 d - T_0 x\| \leq \|T_0\| \cdot \|d - x\| < r \cdot \frac{\varepsilon}{2r} = \frac{\varepsilon}{2}$, so $T_0 \in [d \mapsto B_{\varepsilon/2}(T_0 x)]$ by (1).

Also, for any $T \in [d \mapsto B_{\varepsilon/2}(T_0 x)]$, we have

$$\|Tx - T_0 x\| \leq \|Tx - Td\| + \|Td - T_0 x\| \overset{\text{by (1)}}{\leq} \|T\| \cdot \|d - x\| + \varepsilon/2 < r\frac{\varepsilon}{2r} + \varepsilon/2 = \varepsilon,$$

hence $T \in [x \mapsto B_\varepsilon(T_0 x)]$ by (2). $\square$

Proof via limits. Let $(T_i)_{i \in I} \subseteq \mathcal{B}$ be net and $T \in \mathcal{B}$, and suppose that $T_i d \to Td$ for each $d \in \mathcal{D}$. Fix $x \in X$ in order to show $T_i x \to Tx$. Let $d \in \mathcal{D}$ be such that $\|x - d\| < \frac{\varepsilon}{3r}$. Then

$$\|T_i x - Tx\| \leq \|T_i x - T_i d\| + \|T_i d - Td\| + \|Td - Tx\| \leq \|T_i\| \cdot \|x - d\| + \|T_i d - Td\| + \|T\| \|x - d\|$$
$$\leq r \cdot \frac{\varepsilon}{3r} + \|T_i d - Td\| + r \cdot \frac{\varepsilon}{3r} < \varepsilon \quad \forall^\infty i \text{ because } T_i d \to Td. \quad \square$$

Hilbert spaces.

An inner product on a vector space H is is a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ satisfying:

(i) linearity: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in H$;

(ii) skew-symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$, for all $x, y \in H$;

(iii) positivity: $\langle x, x \rangle \geq 0$, and $x \neq 0 \iff \langle x, x \rangle > 0$.

The space H equipped with an inner product is called an inner product space.

An inner product $\langle \cdot, \cdot \rangle$ on H defines a norm: for each $x \in H$,

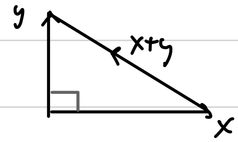
$$\|x\| := \sqrt{\langle x, x \rangle}.$$

This clearly satisfies all norm axioms except the triangle inequality requires proof. We first prove other statements, which are needed to prove the Δ-ineq.

Let H be an inner prod. space. Call vectors $x, y \in H$ orthogonal, write $x \perp y$, if $\langle x, y \rangle = 0$.

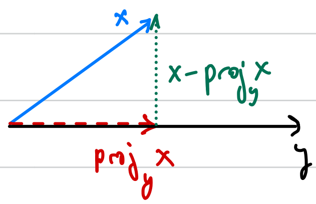
Pythagorean thm. In an inner product space H, if vectors $x \perp y$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Proof. $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle = \|x\|^2 + \|y\|^2$. $\square$



Projections. For vectors $x,y \in H$, where $y \neq 0$, the projection of x onto (the direction of) y is the vector $\mathrm{proj}_y(x) := \langle x, y_1 \rangle y_1$, where $y_1 := \frac{1}{\|y\|} y$.

$$= \frac{\langle x, y \rangle}{\langle y, y \rangle} y$$



Prop. $x - \mathrm{proj}_y x \perp \mathrm{proj}_y x$, equivalently, $x - \mathrm{proj}_y x \perp y$.

Proof. $\langle x - \mathrm{proj}_y x, \mathrm{proj}_y x \rangle = \langle x, \mathrm{proj}_y x \rangle - \|\mathrm{proj}_y x\|^2 = \langle x, \langle x, y_1 \rangle y_1 \rangle - |\langle x, y_1 \rangle|^2 =$

$$= \overline{\langle x, y_1 \rangle} \langle x, y_1 \rangle - |\langle x, y_1 \rangle|^2 = 0. \quad \square$$

Cor. $\|\mathrm{proj}_y x\| \leq \|x\|$.

Proof. $\|x\|^2 = \|\mathrm{proj}_y x + (x - \mathrm{proj}_y x)\|^2 = \|\mathrm{proj}_y x\|^2 + \|x - \mathrm{proj}_y x\|^2 \geq \|\mathrm{proj}_y x\|^2$ by Pythagor. $\square$

Cauchy–Bunyakovski–Schwartz inequality. $|\langle x, y \rangle| \leq \|x\| \|y\|$.

Proof. If $y = 0$, then $\langle x, y \rangle = 0$ because $\langle x, 0 \rangle = \langle x, 0 \rangle + \langle x, 0 \rangle$, and $\|y\| = 0$, so we have nothing to prove. Hence suppose $y \neq 0$. Dividing both sides by $\|y\|$, we may assume $\|y\| = 1$ and we need to prove $|\langle x, y \rangle| \leq \|x\|$. But $\|\mathrm{proj}_y x\| = |\langle x, y \rangle|$, so we've already proved this. $\square$

Δ-inequality. $\|x+y\| \leq \|x\| + \|y\|$.

Proof. $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle = \|x\|^2 + 2\,\mathrm{Re}\langle x,y \rangle + \|y\|^2$

$$\leq \|x\|^2 + 2|\langle x,y \rangle| + \|y\|^2 \underset{CBS}{\leq} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \quad \square$$

Thus $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ is indeed a norm on H, which makes H into a normed vector space. When H is complete, we call it a Hilbert space.

Examples. (a) $\mathbb{C}^n$ is a Hilbert space when equipped with the inner product:

$$\langle x, y\rangle := \sum_{i<n} x(i)\,\overline{y(i)}.$$

Then $\|x\| = \sqrt{\langle x,x\rangle} = \|x\|_2$.

(b) More generally, for any measure space (X,μ), the space $L^2(X,\mu)$ is a Hilbert space when equipped with the inner product:

$$\langle f, g\rangle := \int f\bar{g}\, d\mu.$$

In particular, for any set I, $\ell^2(I)$ is a Hilbert space with the inner product $\langle x,y\rangle := \sum_{i\in I} x(i)\cdot\overline{y(i)}$, and example (a) is just $\ell^2(n) = \mathbb{C}^n$. Again, the norm is just the L^2 norm.

(c) Let $\mathbb{C}[x]$ be the space of all polynomials and view it as $\biguplus_{n\in\mathbb{N}} \mathbb{C}^n$. We equip $\mathbb{C}[x]$ with the following inner product. For $f,g\in\mathbb{C}[x]$, write $f(x) = \sum_{i\le n} a_i x^i$ and $g(x) = \sum_{i\le n} b_i x^i$ for some $n\in\mathbb{N}$, and put: $\langle f,g\rangle := \sum_{i\le n} a_i \bar{b}_i$. This allows for viewing $\mathbb{C}[x]$ as $\biguplus_{n\in\mathbb{N}} \mathbb{C}^n = \biguplus_{n\in\mathbb{N}} \ell^2(n) \subseteq \ell^2(\mathbb{N})$, so $\mathbb{C}[x]$ is a dense subset of $\ell^2(\mathbb{N})$ in the ℓ^2 norm. Thus, $\mathbb{C}[x]$ is not a Hilbert space and $\ell^2(\mathbb{N})$ is its completion.

(d) Let $\mathbb{C}[x]$ be again the space of polynomials, but view it as a subspace of $L^2 := L^2([0,1],\lambda)$. In other words, think of the indeterminant x as taking values in $[0,1]$, which turns polynomials into continuous functions on $[0,1]$. Then $\mathbb{C}[x]$ is equipped with the inner product of L^2, i.e. $\langle f,g\rangle := \int_{[0,1]} f\bar{g}\, d\lambda$. By Weierstrass's thm, polynomials are uniformly dense in $C([0,1])$, which implies that they are also dense in $C([0,1])$ in the L^2 norm (because $[0,1]$ is a finite measure space). We also proved last semester that $C[0,1]$ is dense in L^2 (in the L^2 norm), so $\mathbb{C}[x]$ is dense in L^2. In particular, $\mathbb{C}[x]$ is not a Hilbert space and $L^2([0,1],\lambda)$ is its completion.